J. Fluid Mech. (1969), *vol.* 39, *part* 1, *pp.* 49–55
Printed in Great Britain

Surface tension driven oscillatory instability in a rotating fluid layer

By G. A. McCONAGHY† AND B. A. FINLAYSON
University of Washington Department of Chemical Engineering,
Seattle, Washington 98105

(Received 13 March 1969 and in revised form 16 May 1969)

Oscillatory convective instability is shown to occur in a rotating fluid layer when convection is caused by surface-tension gradients at a free surface. The asymptotic equations, valid when the Taylor number approaches infinity, are solved analytically, and the critical Marangoni number is evaluated numerically. Fluids with Prandtl numbers above 0·201 will exhibit only stationary instability. Fluids with smaller Prandtl numbers will exhibit oscillatory instability with the critical Marangoni number varying as $M_0 T^{\frac{1}{3}}$, where M_0 depends on the Prandtl number and T is the Taylor number.

1. Introduction

The occurrence of cellular convection in fluid layers heated from below can be attributed to two different mechanisms. The buoyancy mechanism, first proposed in Rayleigh's (1916) analysis of Bénard's (1900) experiments, dominates in fluid layers having depths greater than about one centimetre. When the depth of the fluid layer is less than about 0·5 cm the convective motion is predominantly caused by the surface-tension mechanism, which was first analyzed by Pearson (1958). In both cases the convection is inhibited when the fluid layer is rotated about an axis perpendicular to the confining planes. The effect of rotation on buoyancy-driven convection is treated by Chandrasekhar (1953, 1961), Chandrasekhar & Elbert (1955) and Nakagawa & Frenzen (1955) who report that oscillatory instability occurs in addition to stationary instability for fluids having Prandtl number less than 0·677. In contrast, for the surface tension mechanism, Vidal & Acrivos (1966*b*) show that only stationary instability occurs for all Prandtl numbers.

This result is surprising because the usual effect of non-conservative forces (such as the Coriolis force) is to make oscillatory instability possible. This is the case with buoyancy-driven convection when Coriolis or magnetic force terms are included (Chandrasekhar 1961) or in problems of elastic stability (Bolotin 1963). Furthermore, the equations governing either buoyancy or surface-tension driven convection have a remarkable similarity even though in one case the eigenvalue appears in the differential equation whereas in the other case the eigenvalue appears in the boundary conditions. In both cases, however, the introduction of

† Present address: Amoco Chemicals Corp., Whiting, Indiana 46394.

rotational effects leads to identical terms in the equations. Consequently, both the buoyancy and surface-tension driven problems should be affected the same way by the introduction of Coriolis forces.

We can reach the same conclusion on physical grounds if we adopt Veronis's (1959) discussion of the reasons for oscillatory instability. The momentum equation represents a balance of local acceleration, Coriolis, gravitational and viscous forces (for the buoyancy problem). In time-dependent motions of dynamical systems, the local acceleration partially offsets the constraining force of rotation. When the Prandtl number is small, viscous forces become small and the local acceleration becomes more important in the dynamical balance. Consequently we would expect time-dependent motions to be less stable, or more easily generated, since then the local acceleration can offset part of the constraining force of rotation. Time-dependent neutral stability states are of course oscillatory. We would then expect oscillatory instability to occur for buoyancy-driven convection with Coriolis forces. If the convection is driven by surface-tension variations rather than density gradients, the gravitational force does not appear in the momentum balance and the driving force arises in the momentum balance at the free surface. The argument for oscillatory instability did not depend on the source of the driving force, however, and the local acceleration in time-dependent motions should still partially offset the constraining force of rotation. We then expect the same conclusion: oscillatory instability driven by surface-tension variations should be possible in a rotating fluid layer, especially for fluids with small Prandtl numbers.

With these arguments in mind the authors decided to re-examine Vidal & Acrivos's conclusion on the possibility of oscillatory instability in surface-tension driven convection with rotation. If oscillatory instability occurs at all we expect it to occur for large speeds of rotation. Thus the analysis is limited to the case of Taylor number approaching infinity. The analysis is similar to that done by Vidal & Acrivos: the exact solution to the differential equations is substituted into the boundary conditions, leaving a complex number for the eigenvalue, the Marangoni number. Since this number is real (a ratio of physical parameters), solutions can exist only if the imaginary part of the complex number is zero. We search the parameter space numerically for situations where this is true and find that oscillatory instability occurs for values of the parameters not considered by Vidal & Acrivos.

The results are similar in all respects to the buoyancy problem. Stationary instability is the preferred mode of instability if the Prandtl number is above 0·201. For fluids with lower Prandtl numbers, oscillatory instability occurs provided the fluid layer is rotated fast enough.

2. Equations

The linear hydrodynamic stability analysis is similar in both method and notation to that of Chandrasekhar (1961) and Vidal & Acrivos (1966*b*). We consider a physical system consisting of a homogeneous liquid layer having infinite horizontal dimensions and constant depth d. The fluid is confined at the bottom ($z = d$) by an isothermal rigid boundary and at the top ($z = 0$) by a non-deformable free surface, where z is the vertical co-ordinate. The fluid layer is heated uniformly from below while rotating at an angular velocity Ω about the z axis and is assumed to be so shallow that only the surface tension mechanism contributes to instability. We assume that all physical properties of the liquid are constant except surface tension, which varies linearly with temperature.

The basic equations are derived by Vidal & Adrivos (1966*b*); we adopt them with a slight change in non-dimensionalization and assume an exponential time dependence.

$$\left.\begin{aligned} \sigma(D^2-a^2)W &= (D^2-a^2)^2W - T^{\frac{1}{2}}DZ,\\ \sigma P\theta &= (D^2-a^2)\theta - M^{\frac{1}{2}}aW,\\ \sigma Z &= (D^2-a^2)Z + T^{\frac{1}{2}}DW, \end{aligned}\right\} \tag{2.1}$$

where W, θ and Z are the non-dimensional vertical component of velocity, temperature and vertical component of vorticity, measured in units of ν/d, $\beta\nu d/(a\kappa M^{\frac{1}{2}})$, and ν/d^2, respectively. Here ν is the kinematic viscosity, β the temperature gradient, and κ the thermal diffusivity, $P \equiv \nu/\kappa$, and a, the dimensionless wave-number. The boundary conditions are

$$\left.\begin{aligned} &W = DZ = D\theta = D^2W - M^{\frac{1}{2}}a\theta = 0 \quad \text{at} \quad z = 0,\\ &W = Z = \theta = DW = 0 \qquad\qquad\qquad\quad \text{at} \quad z = 1, \end{aligned}\right\} \tag{2.2}$$

where $M \equiv s\beta d^2/\mu\kappa$ is the Marangoni number, $s \equiv d\sigma/dt$ is the surface tension variation with temperature, and μ is the viscosity. Equations (2.1), (2.2) represent a non-self-adjoint eigenvalue problem in the time factor σ. It is well known that such problems admit complex eigenvalues leading to oscillatory instability or decay. Even without rotation, $T = 0$, the problem is non-self-adjoint. In that case, however, oscillatory instability does not occur (Vidal & Acrivos 1966*a*), and it is instructive to rearrange these equations to demonstrate why oscillatory instability should occur for large Taylor number.

We multiply (2.1) by $-W^*$, θ^* and Z^*, respectively, and integrate over the region $0 \leqslant z \leqslant 1$. The asterisk denotes the complex conjugate of a complex number. The boundary conditions are applied and the three equations are then added to obtain

$$\sigma A = -B + aM^{\frac{1}{2}}\left[-\theta(0)DW^*(0) - \int_0^1 \theta^* W\,dz\right] + T^{\frac{1}{2}}2i\,\mathrm{Im}\int_0^1 DZW^*\,dz, \tag{2.3}$$

where A and B are real positive numbers and the coefficient of $T^{\frac{1}{2}}$ is a purely imaginary number. We now consider the asymmetry of (2.3) and argue by analogy with the buoyancy-driven problem. In that case M is replaced by R, the Rayleigh number, and the coefficient of $M^{\frac{1}{2}}$ becomes a real number.† In the buoyancy-driven problem, therefore, when $T = 0$ the time factor σ can only be a real

† Equations in a slightly different form were developed by Finlayson (1968, equation (2.3)) for the buoyancy-driven problem. Those equations, derived for the first approximation when using the Galerkin method, can be reinterpreted in terms of the exact solution and added to obtain the analogue to equation (2.3) here.

number and oscillatory instability is impossible. When $T \neq 0$ complex time factors σ are then possible and are associated with the presence of the term multiplied by $T^{\frac{1}{2}}$.

With the surface-tension driven problem the coefficient of $M^{\frac{1}{2}}$ is no longer a real number and complex time factor σ may be possible even when $T = 0$. Numerical computations of Vidal & Acrivos (1966*a*) indicate, however, that oscillatory instability does not occur when $T = 0$. If rotation is introduced the term multiplying $T^{\frac{1}{2}}$ becomes important as in the buoyancy-driven problem and it would be surprising if oscillatory instability could not occur if T is large enough. In the next section we find that oscillatory instability can indeed occur.

3. Solution

Vidal & Acrivos (1966*b*) have shown that, in the asymptotic case $T \to \infty$, equations (2.1) become independent of Taylor number after the transformation:

$$\left.\begin{aligned} a &= \alpha_0 T^{\frac{1}{4}}, \quad i\sigma = i\gamma T^{\frac{1}{2}}, \quad D = D_0 T^{\frac{1}{4}}, \\ M &= M_0 T^{\frac{1}{2}}, \quad z = zT^{-\frac{1}{4}}, \quad W = W_0, \quad \theta = \theta_0. \end{aligned}\right\} \tag{3.1}$$

Only the boundary conditions at $z = 0$ are important since the solution is confined to an Ekman layer near the free surface. The solution is written as a sum of exponentials whose coefficients are determined to satisfy the boundary conditions. The final expression for the Marangoni number can be written

$$M_0 = \frac{-q \sum_{j=1}^{3} A_j \mu_j^2}{\alpha_0^2 \sum_{j=1}^{3} A_j/(\mu_j + q)}, \tag{3.2}$$

where q is the root of
$$q^2 - \alpha_0^2 - i\gamma P = 0, \tag{3.3}$$

having a negative real part and μ_j are the three roots of

$$(\mu_i^2 - \alpha_0^2)(\mu_j^2 - \alpha_0^2 - i\gamma)^2 + \mu_j^2 = 0, \tag{3.4}$$

having negative real parts, and the A_j are determined by applying the boundary conditions (2.2*a*). Equation (3.2) can be written simply

$$M_0 = RM(\alpha_0, P, \gamma) + i\,\mathrm{Im}\,(\alpha_0, P, \gamma). \tag{3.5}$$

where $RM(\alpha_0, P, \gamma)$ and $\mathrm{Im}\,(\alpha_0, P, \gamma)$ are real valued functions. Since M_0 is a real number we must determine which values of α_0, P and γ make $\mathrm{Im}\,(\alpha_0, P, \gamma) = 0$.

If the principle of exchange of stabilities is valid, only the value $\gamma = 0$ will make $\mathrm{Im}\,(\alpha_0, P, \gamma)$ vanish. The neutral stationary stability curve thus obtained from (3.5) will be a curve of $M_0(\alpha_0)$ which is independent of the Prandtl number. If oscillatory instability occurs, a family of neutral oscillatory stability loci will exist which depend on the Prandtl number.

The neutral oscillatory stability locus will be characterized by non-zero frequency factors γ which make $\mathrm{Im}\,(\alpha_0, P, \gamma)$ vanish for a given Prandtl number and wave-number. The oscillatory neutral stability loci, M_0 *vs.* α_0, have been calculated numerically for Prandtl number = 0·01, 0·10, 0·15, 0·20 and 0·25 and are presented in figure 1. The critical Marangoni and wave-numbers are reported in table 1 and result from numerical minimization of (3.5) subject to the condition that Im (α_0, P, γ) vanish.

Nature of instability	Prandtl number P	Critical frequency factor γ_c	Critical Marangoni number M_0^c	Critical wave-number α_0^c
Stationary	All	0	4·42	0·50
Oscillatory	0·010	0·6887	1·254	0·135
Oscillatory	0·10	0·3446	3·149	0·285
Oscillatory	0·15	0·2434	3·823	0·315
Oscillatory	0·20	0·1477	4·405	0·335
Oscillatory	0·2014	0·1456	4·420	0·336

TABLE 1. Critical values of Marangoni number for oscillatory instability

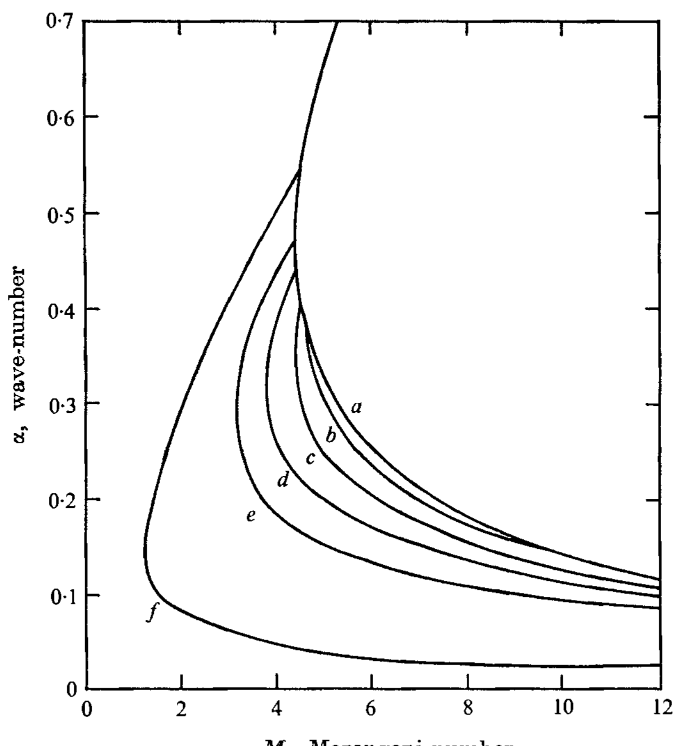

FIGURE 1. Neutral stability curves for stationary and oscillatory instability. *a*, stationary instability; oscillatory instability: *b*, $P = 0·25$; *c*, $P = 0·20$; *d*, $P = 0·15$; *e*, $P = 0·10$; *f*, $P = 0·01$.

A Prandtl number P^* exists such that for all $P > P^*$ the critical Marangoni number for oscillatory instability is always greater than the critical Marangoni number for stationary instability. Hence, for $P > P^*$, convection will always occur as stationary convection and we find, numerically, that $P^* = 0·201$.

These results are qualitatively similar to results for buoyancy-driven convection with Coriolis force. Figure 1 for the surface-tension mechanism is similar to figure 27 in Chandrasekhar (1961, p. 117) for the buoyancy mechanism. Figure 2 summarizes the state of knowledge about the inhibiting effect of Coriolis

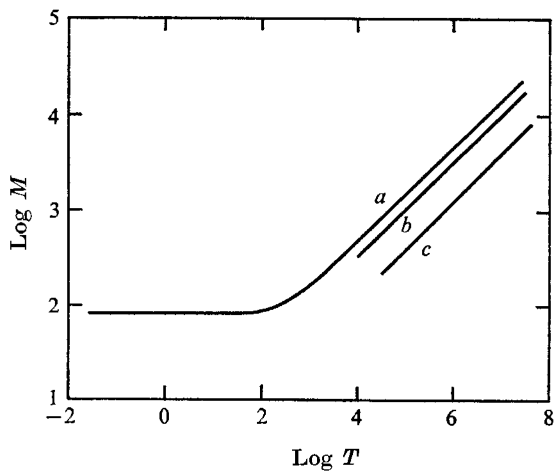

FIGURE 2. Critical Marangoni number as a function of Taylor number. *a*, stationary instability; oscillatory instability: *b*, $P = 0{\cdot}1$: *c*, $P = 0{\cdot}01$.

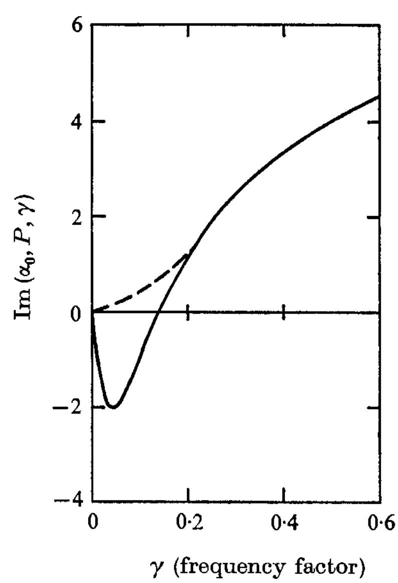

FIGURE 3. The function $\text{Im}(\alpha_0, P, \gamma)$ *versus* frequency factor. $\alpha_0 = 0{\cdot}1$, $P = 0{\cdot}1$ ——, correct value; – – –, incorrect extrapolation.

forces on surface-tension driven convection. Even though results are not known for intermediate Taylor numbers, the behaviour as $T \to \infty$ is similar to that obtained in the buoyancy problem (Chandrasekhar 1961, p. 121). Both mechanisms have a value of P above which no oscillatory instability occurs: $P^* = 0{\cdot}201$

for the surface-tension mechanism whereas $P^* = 0{\cdot}677$ for the buoyancy mechanism. The asymptotic dependence on Taylor number is different since $R = R_0 T^{\frac{2}{3}}$, $\alpha = \alpha_0 T^{\frac{1}{6}}$ for the buoyancy mechanism. For the surface-tension problem the typical results are deduced from table 1: for $P = 0{\cdot}10$, $M = M_0 T^{\frac{1}{2}} = 3{\cdot}149 T^{\frac{1}{2}}$, $a = \alpha_0 T^{\frac{1}{4}} = 0{\cdot}285 T^{\frac{1}{4}}$.

The results presented here differ from those reported by Vidal & Acrivos (1966*b*). Since the two methods of solution are the same, it is important to understand why different results are achieved. These results do not contradict the earlier ones provided we limit consideration to the same parameter values. Vidal & Acrivos calculated the function $\mathrm{Im}\,(\alpha_0, P, \gamma)$ for values of $P = 0{\cdot}1$, $0{\cdot}25$, $0{\cdot}5$, $0{\cdot}7$, 1, 7 and $\alpha_0 = 0{\cdot}1$, $0{\cdot}5$, 1, 2, 5, 10. Of the 36 possible pairs of parameters we find only two yield $\mathrm{Im}\,(\alpha_0, P, \gamma) = 0$ and then only for values of γ not considered previously. For example, for $P = 0{\cdot}1$, $\alpha_0 = 0{\cdot}1$, $\mathrm{Im} = 0$ for $\gamma = 0{\cdot}13$. This root was missed in the earlier work since it examined only $\gamma = 0$, $0{\cdot}2$ and higher values. Figure 3 illustrates the true value of Im (solid line) and the incorrect extrapolation (dotted line) based on values of $\gamma = 0$, $0{\cdot}2$, $0{\cdot}4$, etc.

4. Conclusions

The principle of exchange of stabilities is not valid for surface-tension driven instability with rotation. In the asymptotic limit $T \to \infty$ the conditions are determined for which overstable oscillations will occur. If the Prandtl number is above $0{\cdot}201$ stationary instability is preferred. For smaller Prandtl numbers, the critical Marangoni number for oscillatory instability varies as $M = M_0 T^{\frac{1}{2}}$, where M_0 depends on the Prandtl number.

REFERENCES

Bénard, H. 1900 *Rev. Gen. Sci. Pures Appl., Bull. Assoc. Franc. Avan. Sci.* **11**, 1261, 1309.

Bolotin, V. V. 1963 *Nonconservative Problems of the Theory of Elastic Stability*. London: Macmillan.

Chandrasekhar, S. 1953 *Proc. Roy. Soc. (London)*, A **217**, 306.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press.

Chandrasekhar, S. & Elbert, D. D. 1955 *Proc. Roy. Soc.* A **231**, 198.

Finlayson, B. A. 1968 *J. Fluid Mech.* **33**, 201.

Nakagawa, Y. & Frenzen, P. 1955 *Tellus*, **7**, 1.

Pearson, J. R. A. 1958 *J. Fluid Mech.* **4**, 489.

Rayleigh, Lord 1916 *Phil. Mag.* **32**, 529.

Veronis, G. 1959 *J. Fluid Mech.* **5**, 401.

Vidal, A. & Acrivos, A. 1966*a* *Phys. Fluids*, **11**, 615.

Vidal, A. & Acrivos, A. 1966*b* *J. Fluid Mech.* **26**, 807.